Complex tori

Elliptic curves

Moduli spaces

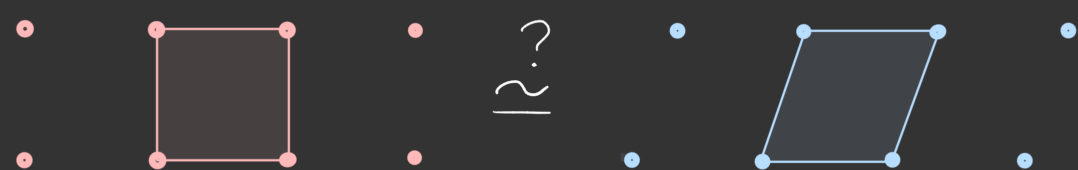
Recall:

Complex tori

Let $\Lambda = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$ be a lattice in $\mathbb{C}$.

A complex torsus is a quotient of the complex plane by a lattice $\mathbb{C}/\Lambda$.

A natural question: classify complex tori as Riemann surfaces.



$\mathbb{C}/\Lambda_1 \overset{?}{\simeq} \mathbb{C}/\Lambda_2$

Proposition: Suppose $\varphi: \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is a holomorphic map between complex tori. Then there exist complex numbers m, b with $m\Lambda_1 \subseteq \Lambda_2$ such that

$$\varphi(z+\Lambda_1) = mz + b + \Lambda_2.$$

The map is invertible if and only if $m\Lambda_1 = \Lambda_2$.

Proof:

Let $\tilde{\varphi}: \mathbb{C} \to \mathbb{C}$ be a lift of φ:

$$\tilde{\varphi}(z) = \varphi(z) \pmod{\Lambda_2}$$

We have

$$\tilde{\varphi}(z+\lambda) = \tilde{\varphi}(z) + \mu_\lambda, \quad \lambda \in \Lambda_1, \; \mu_\lambda \in \Lambda_2.$$

Therefore $\frac{d}{dz}\tilde{\varphi}(z+\lambda) = \frac{d}{dz}\tilde{\varphi}(z)$ for all $\lambda \in \Lambda_1$.

By Liouville's theorem $\frac{d}{dz}\tilde{\varphi}(z) = \text{const}$.

Therefore $\tilde{\varphi}(z) = mz + b$ for some $m, b \in \mathbb{C}$.

$m(z+\lambda) + b \equiv mz + b \pmod{\Lambda_2}$ for all $\lambda \in \Lambda_1$

implies $m\Lambda_1 \subseteq \Lambda_2$. $\blacksquare$

Lemma: For each complex torus $\mathbb{C}/\Lambda$ there exists a unique $\tau \in PSL_2(\mathbb{Z})\setminus h$ such that $\mathbb{C}/\Lambda \simeq \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$

Proof: Let $\Lambda = \lambda_1\mathbb{Z} + \lambda_2\mathbb{Z}$, $\lambda_1, \lambda_2 \in \mathbb{C}$, $\dim_{\mathbb{R}}(\mathbb{R}\lambda_1 + \mathbb{R}\lambda_2) = 2$.

$$\lambda_1/\lambda_2 \in \mathbb{C}\setminus\mathbb{R} = h_+ \sqcup h_-$$

upper half-plane

lower half-plane

Either λ_1/λ_2 or λ_2/λ_1 belongs to h.

$$\lambda_1\mathbb{Z} + \lambda_2\mathbb{Z} = \lambda_1 \cdot \left(\mathbb{Z} + \frac{\lambda_2}{\lambda_1}\mathbb{Z}\right)$$

Therefore $\mathbb{C}/(\lambda_1\mathbb{Z}+\lambda_2\mathbb{Z}) \simeq \mathbb{C}/\left(\mathbb{Z}+\frac{\lambda_2}{\lambda_1}\mathbb{Z}\right)$ as a Riemann surface.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $\tau \in \mathfrak{h}$

$$\mathbb{Z} + \tau\mathbb{Z} = (a\tau+b)\mathbb{Z} + (c\tau+d)\mathbb{Z} =$$

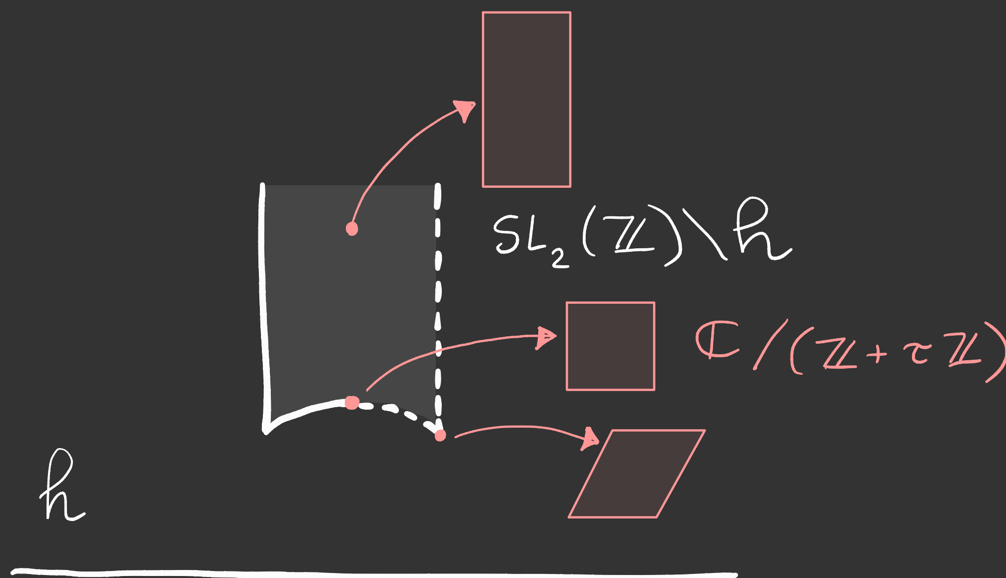
$$= (c\tau+d)\cdot\left(\mathbb{Z} + \frac{a\tau+b}{c\tau+d}\mathbb{Z}\right).$$

Therefore

$$\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \simeq \mathbb{C}/\left(\mathbb{Z} + \frac{a\tau+b}{c\tau+d}\mathbb{Z}\right) \text{ as a Riemann surface.}$$

Claim: Let $\tau_1, \tau_2 \in \mathfrak{h}$. Then $\mathbb{C}/(\mathbb{Z}+\tau_1\mathbb{Z}) \simeq \mathbb{C}/(\mathbb{Z}+\tau_2\mathbb{Z})$ as a RS if and only if $\tau_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau_2$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Moduli space for complex tori



Modular forms

Let $\mathcal{L}$ be the set of lattices in $\mathbb{C}$.

Def: A function $f : \mathcal{L} \to \mathbb{C}$ has weight k iff

even integer

$$f(a \cdot \Lambda) = a^{-k} f(\Lambda), \quad \Lambda \in \mathcal{L}, \quad a \in \mathbb{C}.$$

Then for $\tau \in \mathfrak{h}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$$f(\mathbb{Z} + \tau\mathbb{Z}) = f\big((a\tau+b)\mathbb{Z} + (c\tau+d)\mathbb{Z}\big) =$$

$$= (c\tau+d)^{-k} f\left(\mathbb{Z} + \frac{a\tau+b}{c\tau+d}\mathbb{Z}\right).$$

possibly non-holomorphic functions that transform like

Functions of weight k on $\mathcal{L}$ $\overset{\sim}{\longleftrightarrow}$ Modular forms of weight k on $\mathfrak{h}$

Elliptic functions

Def: A function $f : \mathbb{C} \to \mathbb{C}$ is elliptic with respect to Λ iff

- f is meromorphic on $\mathbb{C}$
- f is Λ-periodic:

$$f(z+\lambda) = f(z) \quad \text{for all} \quad z \in \mathbb{C},\ \lambda \in \Lambda$$

In other words: $f : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ is a meromorphic function.

Notation: $E(\Lambda) = \mathbb{C}(\mathbb{C}/\Lambda)$.

Weierstrass function

Def: $\wp_\Lambda(z) := z^{-2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left((z-\lambda)^{-2} - \lambda^{-2} \right)$

Lemma: $\wp_\Lambda$ is an elliptic function with respect to Λ.

Proof:

Lemma:

- $\wp : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ has exactly one pole. This pole is at the point $0 \pmod{\Lambda}$ and it has order 2.
- $\deg(\wp) = 2$
- $\wp(z_1) = \wp(z_2) \iff z_1 = \pm z_2 \pmod{\Lambda}$.
- The ramification points of Λ are the 4 points of $\frac{1}{2}\Lambda \pmod{\Lambda}$

  Let $\Lambda = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}$. The ramification pts are

  $$0, \quad \frac{\lambda_1}{2}, \quad \frac{\lambda_2}{2}, \quad \frac{\lambda_1+\lambda_2}{2} \mod \Lambda.$$

- $e_1 := \wp\left(\frac{\lambda_1}{2}\right)$, $e_2 := \wp\left(\frac{\lambda_2}{2}\right)$, $e_3 := \wp\left(\frac{\lambda_1+\lambda_2}{2}\right)$

  e_1, e_2, e_3 are distinct complex numbers.

Lemma: The Weierstrass function generates the subfield of all even elliptic functions in $E(\Lambda)$:

$$E(\Lambda)_{\text{even}} = \mathbb{C}(\wp).$$

Proof: Let $f \in E(\Lambda)_{\text{even}}$.

Consider a function

$$g(z) := \prod_{\substack{w \in (\mathbb{C}/\Lambda)/\{\pm 1\} \\ w \neq 0 \bmod \Lambda}} \left(\wp(z) - \wp(w)\right)^{\operatorname{ord}_w(f) \cdot \frac{1}{\operatorname{mult}_w(\wp)}}.$$

Then the function $\frac{f}{g}$ has neither zeroes, nor poles. Therefore $\frac{f}{g} = \text{const} \Rightarrow f \in \mathbb{C}(\wp)$ $\blacksquare$

The function $\wp'(z) = -2\sum_{\lambda \in \Lambda} (z-\lambda)^{-3}$ is odd.

Lemma: The field of all elliptic functions for a lattice Λ is generated by $\wp$ and $\wp'$:

$$E(\Lambda) = \mathbb{C}(\wp, \wp').$$

$\wp$ and $\wp'$ are linearly independent.

Q: Are there algebraic relations between $\wp$ and $\wp'$?

Recall: $$\wp'(z) = -2\sum_{\lambda\in\Lambda} (z-\lambda)^{-3}.$$

The divisor of $\wp'$ is :

$$-3[0] + \left[\tfrac{\lambda_1}{2} \bmod \Lambda\right] + \left[\tfrac{\lambda_2}{2} \bmod \Lambda\right] + \left[\tfrac{\lambda_1+\lambda_2}{2} \bmod \Lambda\right].$$

$(\wp')^2$ is even and therefore belongs to $\mathbb{C}(\wp)$.

$$(\wp'(z))^2 \doteq \left(\wp(z)-\wp\left(\tfrac{\lambda_1}{2}\right)\right)\left(\wp(z)-\wp\left(\tfrac{\lambda_2}{2}\right)\right)\left(\wp(z)-\wp\left(\tfrac{\lambda_1+\lambda_2}{2}\right)\right)$$

up to a multiplicative constant

Q: How to find this constant?

Eisenstein series & Laurent expansion of $\wp$

Def: Eisenstein series $G_k(\Lambda) := \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-k}$

$k \geq 4$ even.

Lemma: The Laurent expansion of $\wp$ is

$$\wp(z) = z^{-2} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} (n+1) G_{n+2}(\Lambda) z^n$$

for all z such that $0 < |z| < \inf\{|w| : w \in \Lambda \setminus \{0\}\}$

Proof:

$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \sum_{n=1}^{\infty} (n+1) z^n \cdot \lambda^{-n-2}$$

Lemma: $(\wp'(z))^2 = 4(\wp(z))^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$.

where $g_2(\Lambda) = 60\,G_4(\Lambda)$ and $g_3(\Lambda) = 140\,G_6(\Lambda)$.

Proof:

$$(\wp'(z))^2 = \frac{4}{z^6} - 24\frac{G_4(\Lambda)}{z^2} - 80\,G_6(\Lambda) + O(z^2)$$

$$4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda) =$$

Therefore the elliptic function

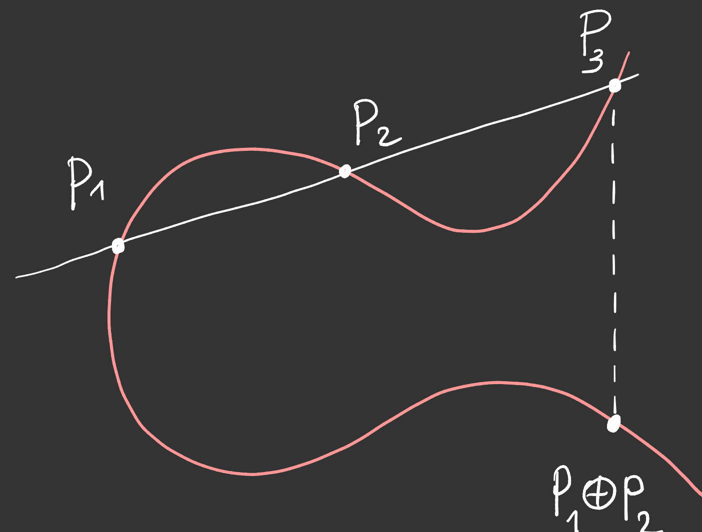
$$(\wp'(z))^2 - \left(4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)\right)$$

has no poles and has a zero at $z=0$.

Thus, this function is identically zero $\blacksquare$

$(\wp, \wp')$: Complex torus $\longrightarrow$ Elliptic curve

$\mathbb{C}/\Lambda$ $\qquad$ $E: y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$



Lemma: The points $(\wp(z_1), \wp'(z_1))$, $(\wp(z_2), \wp'(z_2))$, $(\wp(z_3), \wp'(z_3))$ are collinear if and only if $z_1 + z_2 + z_3 = 0 \mod \Lambda$.

Def: An enhanced elliptic curve for $\Gamma_0(N)$ is an ordered pair (E, C) where E is a complex elliptic curve and C is a cyclic subgroup of E of order N.

Two such pairs (E, C) and (E', C') are equivalent if some isomorphism $E \xrightarrow{\sim} E'$ takes C to C'.

$\Gamma_0(N) \backslash \mathfrak{h}$ $\underset{1:1}{\overset{\sim}{\longleftrightarrow}}$ Equivalence classes of enhanced elliptic curves for $\Gamma_0(N)$

Def: An enhanced elliptic curve for $\Gamma_1(N)$ is an ordered pair (E, Q) where E is a complex elliptic curve and Q point of E of order N.

Two such pairs (E, Q) and (E', Q') are equivalent if some isomorphism $E \xrightarrow{\sim} E'$ takes Q to Q'.

Def: An *enhanced elliptic curve for* $\Gamma(N)$ is an ordered pair $(E,(P,Q))$ where E is a complex elliptic curve and (P,Q) is a pair of points of E that generates the N-torsion subgroup $E[N]$ with Weil pairing $e_N(P,Q) = e^{2\pi i/N}$

Two such pairs $(E,(P,Q))$ and $(E',(P',Q'))$ are *equivalent* if some isomorphism $E \xrightarrow{\sim} E'$ takes (P,Q) to (P',Q').